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ON THE GROWTH OF RANDOM KNAPSACKS

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ABSTRACT

We considers the problem of optimally filling a knapsack of fixed capacity by choosing from among a collection of n objects of randomly determined weight and value. Under very mild conditions on the common joint distribution of weight and value, we determine the asymptotic value of the optimal knapsack, for large n.

1 INTRODUCTION

The random version of the classical single-constraint, 0–1 linear programming problem is given by

$$\begin{aligned} V_n = \quad & \max \quad \sum_{i=1}^{n} X_i \delta_i \qquad\qquad (1.1) \\ & \text{subject to} \quad \sum_{i=1}^{n} W_i \delta_i \leq 1 \quad \delta_i \in \{0,1\} \end{aligned}$$

where the pairs (W_i, X_i) are assumed to be independent draws from a common joint distribution F_{WX}. If we think of the pairs (W_i, X_i) as the weights and values, respectively, of a collection of n objects, then this problem can be thought of as finding the collection of objects of maximum value which will fit in a "knapsack" with weight capacity 1.

In this paper we shall compute the asymptotic value of the random variables V_n with increasing n, for a very large class of joint distributions F_{WX}.

Frieze and Clarke (1984) computed the asymptotic value of this random knapsack problem for a particular F_{WX} (where W and X are mutually independent and both uniformly distributed on the interval (0,1)) as part of their analysis of approximation algorithms for the deterministic knapsack problem. In a related paper, Meante, Rinnooy Kan, Stougie, and Vercellis (1984) analyze a random knapsack problem in which the knapsack capacity grows in proportion to the number n of items. Under this assumption, the question as to the growth rate of V_n is easily resolved—the strong law of large numbers guarantees that V_n/n converges. Among their results, Meante *et al* compute this limit. In contrast, when the knapsack capacity is fixed the growth rate of V_n depends on the joint distribution of weights and values. This dependence is our main interest here.

In section 2 we state and prove our main result (Theorem 1) characterizing the asymptotic value of V_n. In section 3 we present some examples. In one of them (Example 2) we apply Theorem 1 to a multiconstraint knapsack problem of Frieze and Clarke (1984) to obtain a new asymptotic upper bound for the value of this problem. In fact, our upper bound turns out to be the asymptotic value (Schilling (1988)). We also consider a class of examples in which weight and value are functionally related.

2 RESULTS

We define V_n as in (1.1). Our goal is to compute the asymptotic value of V_n as $n \to \infty$. Let us begin with 2 easy cases.

(i) Suppose that $P(W = 0 \text{ and } X > 0) > 0$. Then V_n grows linearly in n. To prove this, let $X_i' = S_i \cdot 1_{\{W_i=0\}}$. Then $E(X') = \lim_{n\to\infty} \frac{X_1' + \cdots + X_n'}{n} \leq \liminf_{n\to\infty} \frac{V_n}{n}$ $\leq \limsup_{n\to\infty} \frac{V_n}{n} \leq \lim_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = E(X)$ a.s. (The equalities follow from the strong law of large numbers). Thus we shall require that $W > 0$.

(ii) Suppose that, for some t, $\frac{X}{W} \leq t$ a.s. Then, for all n, $V_n \leq t$; V_n doesn't grow at all. Thus we shall require that $\frac{X}{W}$ be unbounded.

We now officially assume the common joint distribution F_{WX} of the pairs (W_i, X_i) satisfies

$$\begin{array}{l} W > 0 \text{ and } 0 < X < 1, \text{ the random variable } \frac{X}{W} \text{ is continuous,} \\ \text{and its density } f_{\frac{X}{W}}(t) \text{ is positive for all sufficiently large } t. \end{array} \tag{A1}$$

(We shall also require a "regularity" assumption (A2), which we postpone stating until after lemma 1 when we will have had a chance to motivate it.)

DEFINITIONS: Let (Z_n) be a sequence of random variables, and (x_n) a sequence of numbers. We write $Z_n \sim x_n$ to mean that $P(x_n(1 - o(1)) \leq Z_n \leq x_n(1 + o(1))) \to 1$ as $n \to \infty$. As usual, o(1) denotes a sequence that converges to 0 as $n \to \infty$. This is the standard notion of "almost everywhere" for random combinatorics (as in Bollabas (1985), e.g.).

$$\text{For } t > 0, \text{ let } F(t) = E(W \cdot 1_{\{\frac{X}{W} \geq t\}}), \text{ and } G(t) = E(X \cdot 1_{\{\frac{X}{W} \geq t\}}).$$

Our analysis begins by collecting some simple facts about the functions F and G.

Lemma 1 *(i) For all sufficiently large t, $F(t)$ and $G(t)$ are continuous and monotone decreasing, and $\lim_{t\to\infty} F(t) = \lim_{t\to\infty} G(t) = 0$.*

(ii) $F^{-1}(t)$ exists for all sufficiently small positive t, and $\lim_{t\to 0^+} F^{-1}(t) = \infty$.

(iii) $G \circ F^{-1}(t)$ is monotone increasing on some interval $(0, \varepsilon)$, and $\lim_{t\to 0^+} G \circ F^{-1}(t) = 0$.

(iv) $\lim_{t\to 0^+} \frac{G \circ F^{-1}(t)}{t} = \infty$.

Proof: From assumption (A1) we can see that $P(\frac{X}{W} \geq 1)$ decreases montonically to 0 as $t \to \infty$. (i) follows almost at once, then (ii) follows from (i), and (iii) from (i) and (ii). To prove (iv), note that

$$G(u) = E(X \cdot 1_{\{\frac{X}{W} \geq u\}}) \geq E(uW \cdot 1_{\{\frac{X}{W} \geq u\}}) = uF(u).$$

Let $u = F^{-1}(t)$; then we have

$$G \circ F^{-1}(t) \geq F^{-1}(t)t, \text{ i.e. } \frac{G \circ F^{-1}(t)}{t} \geq F^{-1}(t),$$

and so (iv) follows from (ii). ■

In light of (iii), we let $G \circ F^{-1}(0) = 0$. Thus $G \circ F^{-1}$ is continuous at 0, increasing to the right of 0, and has infinite derivative at 0. Any reasonable function with these properties will be concave (i.e., lie above its chords) on some interval $[0, \varepsilon]$.

We shall assume from here on

$(A2)$ for some $\varepsilon > 0$, $G \circ F^{-1}$ is concave on the interval $[0, \varepsilon]$.

The purpose of (A2) is to facilitate the proof of the following technical lemma.

Lemma 2 *There exist sequences (t_n), (u_n) of positive numbers with the following properties:*

(i) For all n, $F(t_n) > \dfrac{1}{n} > F(u_n)$.

(ii) As $n \to \infty$, $nF(t_n) \to 1$ and $nF(u_n) \to 1$.

(iii) As $n \to \infty$, $t_n(1 - nF(t_n))^2 \to \infty$ and $u_n(1 - nF(u_n))^2 \to \infty$.

(iv) As $n \to \infty$, $\dfrac{G(t_n)}{G \circ F^{-1}(\frac{1}{n})} \to 1$ and $\dfrac{G(u_n)}{G \circ F^{-1}(\frac{1}{n})} \to 1$.

Proof: Note that, for all $\varepsilon > 0$, by Lemma 1(ii), for all sufficiently large n, $F^{-1}(\frac{1}{n}(1+\varepsilon)) \cdot \varepsilon^2 > \frac{1}{\varepsilon}$. Let $N_\varepsilon = \min\{n : F^{-1}(\frac{1}{n}(1+\varepsilon)) \cdot \varepsilon^2 > \frac{1}{\varepsilon}\}$. Now $N_\varepsilon \to \infty$ as $\varepsilon \to 0^+$, so there exists a sequence (δ_n) of positive numbers such that $\delta_n \to 0$ as $n \to \infty$, and $N_{\delta_1} < N_{\delta_2} < N_{\delta_3} < \ldots$. Finally, let

$$\varepsilon_n = \begin{cases} 1 & \text{if } n < N_{\delta_1} \\ \delta_1 & \text{if } N_{\delta_1} \leq n < N_{\delta_2} \\ \delta_2 & \text{if } N_{\delta_2} \leq n < N_{\delta_3} \\ \text{etc.} & \end{cases}$$

Now it is easy to check that $\varepsilon_n \to 0$ and $F^{-1}(\frac{1}{n}(1+\varepsilon_n)) \cdot \varepsilon_n^2 \to \infty$ as $n \to \infty$. Finally, let $t_n = F^{-1}(\frac{1}{n}(1+\varepsilon_n))$. The parts of i–iii having to do with t follow easily. To verify the t-part of iv, we require a sublemma.

Sublemma 3 *Suppose that H is a non-negative, concave function on $[0,\varepsilon]$, and $H(0)=0$. Then if $a>1$ and $ax \in [0,\varepsilon]$, then $\dfrac{H(ax)}{H(x)} < a$. If $0<a<1$, then $\dfrac{H(ax)}{H(x)} > a$.*

Suppose $a>1$, then $ax > x$, and because of the concavity of H we have

$$\frac{H(ax)-H(x)}{ax-x} < \frac{H(x)}{x}.$$

Therefore,

$$\frac{H(ax)}{H(x)} = 1 + (a-1)\cdot\frac{H(ax)-H(x)}{ax-x}\cdot\frac{x}{H(x)} < 1 + (a-1) = a.$$

The second assertion follows since $0<a<1$ implies

$$\frac{H(x)-H(ax)}{x-ax} < \frac{H(ax)}{ax}.$$

Now apply sublemma 3 with $H = G\circ F^{-1}$, $x=\frac{1}{n}$, and $a = 1+\varepsilon_n$. We have, for all large enough n,

$$\frac{G(t_n)}{G\circ F^{-1}(\frac{1}{n})} = \frac{G\circ F^{-1}(\frac{1}{n}(1+\varepsilon_n))}{G\circ F^{-1}(\frac{1}{n})} < 1+\varepsilon_n.$$

On the other hand, since $G\circ F^{-1}$ is increasing.

$$\frac{G\circ F^{-1}(\frac{1}{n}(1+\varepsilon_n))}{G\circ F^{-1}(\frac{1}{n})} \geq 1,$$

and the t-part of (iv) follows. The u-parts of lemma 2 are proved by replacing $1+\varepsilon$ everywhere by $1-\varepsilon$. ■

We now state and prove our main result:

Theorem 1 *$V_n \sim n\cdot G\circ F^{-1}(\frac{1}{n})$, where V_n is as defined as in (1.1).*

<u>Proof</u>:

$$\begin{aligned}\text{For } t>0 \text{ let } \hat{W}_n(t) &= \sum_{i=1}^{n} W_i \cdot 1_{\{\frac{X_i}{W_i}\geq t\}} \\ \text{and } \hat{X}_n(t) &= \sum_{i=1}^{n} X_i \cdot 1_{\{\frac{X_i}{W_i}\geq t\}}.\end{aligned}$$

Now it is easy to compute $E(\hat{W}_n(t)) = nF(t)$ and

$$\begin{aligned}
\mathrm{Var}(\hat{W}_n(t)) &\le nE(W^2 \cdot 1_{\{\frac{X}{W} \ge t\}}) \\
&\le \tfrac{n}{t} E(W \cdot 1_{\{\frac{X}{W} \ge t\}}) \quad \text{since } X < 1 \text{ and } \tfrac{X}{W} \ge t \text{ imply } W \le \tfrac{1}{t} \\
&= nF(t)/t.
\end{aligned}$$

Let (t_n) be as in lemma 2. We have

$$\begin{aligned}
P(\hat{W}_n(t_n) < 1) &= P(\hat{W}_n(t_n) - E(\hat{W}_n(t_n)) < 1 - nF(t_n)) \\
&\le \frac{nF(t_n)}{t_n(1 - nF(t_n))^2} \text{ (by Chebyshev's inequality).}
\end{aligned}$$

Thus, by lemma 2, $P(\hat{W}_n(t_n) < 1) \to 0$ as $n \to \infty$. By a symmetrical argument, $P(\hat{W}_n(u_n) > 1) \to 0$ as $n \to \infty$. Hence

$$P(\hat{W}_n(t_n) \ge 1 \ge \hat{W}_n(u_n)) \to 1 \text{ as } n \to \infty. \tag{2.1}$$

Next recall the greedy algorithm for problem (1.1). Order the pairs (W_i, X_i) so that $\frac{X_{(1)}}{W_{(1)}} \ge \frac{X_{(2)}}{W_{(2)}} \ge \ldots \ge \frac{X_{(n)}}{W_{(n)}}$. If we let V_n' denote the value obtained in (1.1) by the greedy algorithm, then $V_n' = X_{(1)} + \cdots + X_{(k)}$, where k is the greatest number among $1, \ldots, n$ such that $W_{(1)} + \cdots + W_{(k)} \le 1$. In particular if $\hat{W}_n(t) \ge 1 \ge \hat{W}_n(u)$, then $\hat{X}_n(t) \ge V_n' \ge \hat{X}_n(u)$. Thus from (2.1) we have

$$P(\hat{X}_n(t_n) \ge V_n' \ge \hat{X}_n(u_n)) \to 1 \text{ as } n \to \infty. \tag{2.2}$$

Finally compute,

$$\begin{aligned}
E(\hat{X}_n(t_n)) &= nG(t_n) \text{ and} \\
\mathrm{Var}(\hat{X}_n(t_n)) &\le nE(X^2 \cdot 1_{\{\frac{X}{W} \ge t\}}) \\
&\le nE(X \cdot 1_{\{\frac{X}{W} \ge t\}}) \quad \text{since } X < 1 \\
&= nG(t_n),
\end{aligned}$$

so $\mathrm{Var}(\frac{\hat{X}_n(t_n)}{nG(t_n)}) = 1/nG(t_n)$. Now by lemma 1(iv), as $n \to \infty$, $nG \circ F^{-1}(\frac{1}{n}) \to \infty$, so by lemma 2(iv), $nG(t_n) \to \infty$. Thus by Chebyshev's inequality, $\hat{X}_n(t_n) \sim nG(t_n)$. By another use of lemma 2(iv), we have $\hat{X}_n(t_n) \sim nG \circ F^{-1}(\frac{1}{n})$. By a symmetric argument, applied to u_n, it can be shown that $\hat{X}_n(u_n) \sim nG \circ F^{-1}(\frac{1}{n})$. Thus, by (2.2), $V_n' \sim nG \circ F^{-1}(\frac{1}{n})$. But it is well known that $V_n' \le V_n \le V_n' + 1$; therefore $V_n \sim nG \circ F^{-1}(\frac{1}{n})$, which completes the proof of theorem 1. ∎

3 Examples

Example 1 (Frieze and Clarke, 1984). Consider the knapsack problem (1.1) where W and X are mutually independent and both distributed uniformly of the interval (0,1). For $t \geq 1$, we have

$$F(t) = E(W \cdot 1_{\{\frac{X}{W} \geq t\}}) = \int_{\{(w,x) \in (0,1)^2 : x \geq tw\}} w dw dx = \frac{1}{6t^2},$$

and

$$G(t) = E(X \cdot 1_{\{\frac{X}{W} \geq t\}}) = \int_{\{(w,x) \in (0,1)^2 : x \geq tw\}} x dw dx = \frac{1}{3t}.$$

Therefore, by Theorem 1, $V_n \sim nG \circ F^{-1}(\frac{1}{n}) = \sqrt{\frac{2n}{3}}$. This duplicates a result originally obtained in Frieze and Clarke (1984).

Example 2 (Frieze and Clarke, 1984). Consider the m-dimensional knapsack problem

$$\begin{array}{llll} V_n = & \max & \sum_{j=1}^n X_j \delta_j & \\ & \text{subject to} & \sum_{j=1}^n W_{ij} \delta_j \leq 1 \quad \text{for } i = 1, 2, \ldots, m, & \delta_j \in \{0,1\}. \end{array} \tag{3.1}$$

The W_{ij} and X_i are assumed to be mutually independent, and all uniformly distributed on the interval (0,1). This problem may be compared to two related one-dimensional problems:

$$\begin{array}{lll} \underline{V}_n = & \max & \sum_{j=1}^n X_j \delta_j \\ & \text{subject to} & \sum_{j=1} \underline{W}_j \delta_j \leq 1 \quad \delta_j \in \{0,1\}, \end{array} \tag{3.2}$$

where $\underline{W}_j = \max\{W_{1j}, W_{2j}, \ldots, W_{mj}\}$ for $j = 1, 2, \ldots, n$ and

$$\begin{array}{lll} \overline{V}_n = & \max & \sum_{j=1}^n X_j \delta_j \\ & \text{subject to} & \sum_{j=1}^n \overline{W}_j \delta_j \leq 1 \quad \delta_j \in \{0,1\}, \end{array} \tag{3.3}$$

where

$$\overline{W}_j = \frac{W_{1j} + W_{2j} + \cdots + W_{mj}}{m} \quad \text{for } j = 1, 2, \ldots, n.$$

It is not hard to see that we have $\underline{V}_n \leq V_n \leq \overline{V}_n$. A series of computations, along with Theorem 1, show that

$$\underline{V}_n \sim \left(\frac{(m+1)^m n}{m^m (m+2)} \right)^{\frac{1}{m+1}} \text{ and } \overline{V}_n \sim (m+1) \left(\frac{n}{(m+2)!} \right)^{\frac{1}{m+1}}.$$

Hence

$$P\left(\left(\frac{(m+1)^m n}{m^m(m+2)}\right)^{\frac{1}{m+1}}(1-o(1)) \le V_n \le (m+1)\left(\frac{n}{(m+2)!}\right)^{\frac{1}{m+1}}(1+o(1)\right) \to 1 \tag{3.4}$$

as $n \to \infty$.

The lower bound in (3.4) duplicates a result in Frieze and Clarke (1984). The upper bound in (3.4) is, so far as we know, new. Furthermore, it is sharp. Indeed, it is shown in Schilling (1988) that

$$V_n \sim (m+1)\left(\frac{n}{(m+2)!}\right)^{\frac{1}{m+1}}.$$

Our third example relates the shape of the joint density of W and X near the x-axis to the growth rate of the value of the knapsack.

Example 3: Suppose that for some $\varepsilon > 0$, $f_{WX}(w,x) = \alpha w^{\beta_1} x^{\beta_2}$ on the set $\{(x,w)|\frac{x}{w} > \varepsilon\}$. Then, for $t > \varepsilon$,

$$\begin{aligned} G(t) &= \int\int_{\{(w,x)|\frac{x}{w}\ge t\}} x\alpha w^{\beta_1} x^{\beta_2} dw dx \\ &= \frac{\alpha}{(\beta_1+1)(\beta_1+\beta_2+3)t^{\beta_1+1}} \end{aligned}$$

$$\begin{aligned} F(t) &= \int\int_{\{(w,x)|\frac{x}{w}\ge t\}} \alpha w w^{\beta_1} x^{\beta_2} dw dx \\ &= \frac{\alpha}{(\beta_1+2)(\beta_1+\beta_2+3)t^{\beta_1+2}}. \end{aligned}$$

Thus, by Theorem 1,

$$\begin{aligned} V_n \sim nG\circ F^{-1}(\frac{1}{n}) &= n\left(\frac{\alpha}{\beta_1+\beta_2+3}\right)^{\frac{1}{\beta_1+2}}\left(\frac{(\beta_1+2)^{\frac{\beta_1+1}{\beta_1+2}}}{\beta_1+1}\right)\left(\frac{1}{n}\right)^{\frac{\beta_1+1}{\beta_1+2}} \\ &= \left(\frac{(\beta_1+2)^{\frac{\beta_1+1}{\beta_1+2}}}{\beta_1+1}\right)\left(\frac{\alpha n}{\beta_1+\beta_2+3}\right)^{\frac{1}{\beta_1+2}} \end{aligned}$$

The hypotheses of Theorem 1 do not require that W and X be jointly continuous, only that $\frac{X}{W}$ be a continuous random variable. When weight and value are related in a deterministic fashion, F_{WX} may be singular with respect to Lebesgue measure on $(0,1)^2$, but $\frac{X}{W}$ still nonetheless continuous. Our final example is such a case.

Example 4. Consider the problem

$$V_n = \max \sum_{i=1}^n X_i\delta_i \tag{3.4}$$
$$\text{subject to } \sum_{i=1}^n X_i^\alpha \delta_i \le 1 \quad \delta_i \in \{0,1\},$$

where $\alpha > 1$, and each X_i is uniform on the interval (0,1).

Then

$$F(t) = E(X^\alpha \cdot 1_{\{X^{\alpha-1} \ge t\}}) = \frac{1}{\alpha+1} t^{(\frac{\alpha+1}{\alpha-1})}$$

and

$$G(t) = E(X \cdot 1_{\{X^{\alpha-1} \ge t\}}) = t^{(\frac{2}{\alpha-1})}$$

so, by Theorem 1,

$$V_n \sim \frac{1}{2}(\alpha+1)^{(\frac{2}{\alpha+1})} n^{(\frac{\alpha-1}{\alpha+1})}.$$

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